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Notes for Neil Sloane on his Handbook

1 Abbreviations ICA = Introduction to Comb. Analysis (JR):
C.I = Combinatorial Identities ; seq = number sequence

2.a Seq. 708 is $S_n(1)$: $S_n(x)$ is the square rook polynomial (ICA p. 184) and it might be noticed that

$$S_n(1) = 2n\, S_{n-1}(1) - (n-1)^2 S_{n-2}(1)$$

2b. $S_n'(1) = [(d/dx) S_n(x)]_{x=1}$ has the sequence (not in Handbook)

n	0	1	2	3	4	5	6	7	8	9	10
$S_n'(1)$	0	1	8	63	544	5225	55656	653023	8379008	116780049	1757211400

and $S_n'(1) = n^2 S_{n-1}(1)$

3a. Seq 1190 has the identification Forests of Greatest Height, which is the same as $l_n(1)$: $l_n(x)$ = Lah polynomial = $(-1)^n L_n(x)$ where $L_n(x)$ is identified in ICA, p. 43 & 44. It might be noticed that $l_n(1) = (2n-1)\, l_{n-1}(1) - (n-1)(n-2)\, l_{n-2}(1)$ or
$l_n(1) = S_n(1) - n\, S_{n-1}(1)$ with $S_n(x)$ = square rook polynomial above

3b. The sequence of derivatives $l_n'(1)$ is

n	0	1	2	3	4	5	6	7	8	9	10
$l_n'(1)$	0	1	4	21	136	1045	9276	93289	1047376	12975561	175721140

Also $l_n'(1) = n\, S_{n-1}(1)$, $S_n(x)$ square rook polyn. as above
$l_n'(1)$ is not in the Handbook

4a The subfactorial = rencontres numbers in seq. 766 are also those of $d_n(1)$, where $d_n(x) = \Sigma\, d(n,k)\, x^k$, $d(n,k)$ the associated Stirling numbers of the first kind ICA 73 ; the recurrence
$d_n(1) = (n-1)\,(d_{n-1}(1) + d_{n-2}(1))$ might be noticed.

JR 9/25/80

4b The sequence of derivatives $d_n'(1)$ is

n	0	1	2	3	4	5	6	7	8	9	10
$d_n'(1)$	0	0	1	2	12	64	425	3198	27216	258144	2701737
and $\frac{1}{n-1} d_n'(1)$	0	0	1	1	4	16	64	425	3198	27216	258144

The recurrence for the latter is

$$\delta_{n+1} = \frac{1}{n} d_{n+1}'(1) = (2n-3)\, d_n - (n-2)\left[(n-4)\,\delta_{n-1} + 2(n-3)\,\delta_{n-2} + (n-4)\,\delta_{n-3}\right]$$

5 With $\Delta_n(x) = \sum_0 d_{n+k,k}\, x^k$, and d_{nk} as above, $\Delta_n(x)$ has the recurrence

$$\Delta_n(x) = (n-1+nx)\,\Delta_{n-1}(x) + (x+x^2)\,\Delta_{n-1}'(x)$$ (prime = derivative

Neither of the following sequences is in the Handbook.

n	0	1	2	3	4	5	6	7	8	9
$\Delta_n(1)$	1	1	5	41	469	6889	123605	2620169	64094901	1775623081
$\Delta_n'(1)$	0	1	8	91	1334	23913	506652	12386183	343174882	10626342453

N.B. The numbers $d_{n+k,k}$ appear in Tables of Binomial Coefficients and Stirling Nos, Journal of Research, National Bureau of Standards, B, Math Sciences Vol 80B No 1 March 1976 with $d_{n+k,k} = I_n^{k}$ (pages 155 - 163)

6. The numbers in seq. 1611 are from a paper by W. T. Tutte, in his notation $h(1,n)$ and

$$h(1,n) = \frac{1}{2}\left[4^{n+1} - \binom{2n+2}{n+1}\right]$$

Also if $T_n(x) = \sum a_{n+k,n-k}\, x^k$ with a_{nm} a ballot no. then $h(1,n)$ and $T_n'(1)$ (prime = derivate) are related as follows: $T_n'(1) = 4T_{n-1}'(1) + C_{n-1}$ (C_n - Catalan no) $= h(1,n-1)$

n	0	1	2	3	4	5	6	. .
$h(1,n)$	1	5	22	93	386	1586	6476	
$T_n'(1)$	0	1	5	22	93	386	1586	

9/25/80

4 (cont) Incidentally $T_n(1) = \binom{2n}{n}$ is seg 613 ; $\frac{1}{2}T_n(1)$ is seg 1144

5. In C.I p.66 $p_n(x) = \sum_0 \binom{n+k}{2k} x^k$ and

$p_n(x) - (2+x)p_{n-1}(x) + p_{n-2}(x) = \delta_{n0} - \delta_{n1}$ Kronecker deltas

$p_0(x) = 1$, $p_1(x) = 1+x$; $p_n(x) = (2+x)p_{n-1}(x) - p_{n-2}(x)$ $n = 2, 3, \ldots$

Seq. 569 is $p_n(1)$, seq 1160 is $p_n(2)$, seq. 1630 is $p_n(4)$

The following sequences are interesting candidates (space permitting)

n	0	1	2	3	4	5	6	7	8	9
$p_n(3)$	1	4	14	91	436	2084	10009	47956	~~237641~~ 229771	1100899
$p_n(5)$	1	6	41	281	1926	13201	90481	620166	4250781	

The recurrences are $p_n(3) = 5p_{n-1}(3) - p_{n-2}(3)$ $n = 2, 3, \ldots$

$p_n(5) = 7p_{n-1}(5) - p_{n-2}(5)$, $n = 2, 3, \ldots$

and $p_0(j) = 1$, $p_1(j) = j+1$

Sequence 1595 is $p_n'(1)$!

6. CI p.78 $Q_n(x) = \sum_0^n \binom{n+k}{2k}\binom{2k}{k} x^k$, $Q_0(x) = 1$; $Q_1(x) = 1+2x$

CI p.79 $(n+1)Q_{n+1}(x) - (2n+1)(1+2x)Q_n(x) + nQ_{n-1}(x) = 0$

Seq. 1184 is $Q_n(1)$ whose recurrence is $nQ_n(1) - 3(2n-1)Q_{n-1}(1) + (n-1)Q_{n-2}(1) = 0$

The caption of 1184 – A Square Recurrence indicates a ~~second~~ new significance to $Q_n(x)$! (or perhaps only superficially new).

Strangely $\frac{1}{2}Q_n'(1)$ is seq. 1985 except for $n = 5, 10$!

~~7. C.I 66 , $p_n(x) = \sum_0^n \binom{n+k}{2k} x^k$~~

~~with recurrence $p_n(x) - (2+x)p_{n-1}(x) + p_{n-2}(x) = \delta_{n0} + \delta_{n1}$~~

9/26/80

7. CI 86 $\quad p_n(x) = \Pi_n(x) - \Pi_{n-1}(x)$, $\quad \Pi_n(x) = \sum_0 \binom{n+k+1}{2k+1} x^k$

n	0	1	2	3	4	5	6	7	8	9	10
$\Pi_n(1)$	1	3	8	21	55	144	377	987	2584	6765	17711

This seq. is not in the Handbook. Of course $\Pi_n(1) = p_n(1) + \Pi_{n-1}(1)$

$\Pi_n'(1)$	0	1	6	25	90	300	954	2939	8850	26195	76500

(This sequence is 1733 up to $\Pi_6'(1)$ and 1733 is captioned $\Pi_n'(1)$ in my additions to the Handbook ; so some suspicion must be followed)