THE GOLDEN SECTION, PHYLLOTAXIS, AND WYTHOFF'S GAME

By H. S. M. Coxeter

> Geometry has two great treasures: one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel.
> —J. Kepler (1571–1630)

1. The odd-sounding phrase "division of a line into extreme and mean ratio" was used by Euclid to signify division of a line segment into two unequal parts such that the ratio of the whole to the larger part is equal to the ratio of the larger to the smaller. Calling each ratio τ (after τομή, "the section"), we see that this requires

$$\tau^{-1} + \tau^{-2} = 1,$$

so that τ is the positive root of the equation

$$x^2 - x - 1 = 0,$$

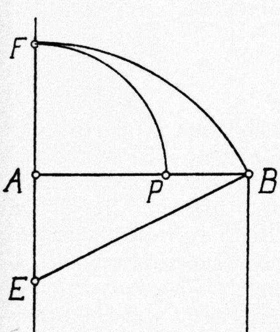

Fig. 1

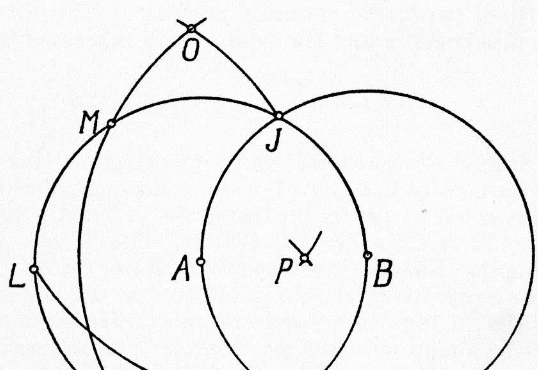

Fig. 2

viz.,

$$\tau = {}^1/_2(\sqrt{5} + 1) = 1.618033989\ldots,$$

whence

$$\tau^{-1} = {}^1/_2(\sqrt{5} - 1), \qquad \tau^{-2} = {}^1/_2(3 - \sqrt{5}).$$

The classical construction (Euclid II, 11) is as follows. To divide a given segment AB in extreme and mean ratio (Fig. 1), let E be the mid-point of the side AC of the square $ABDC$; take F, on CA produced, so that $EF = EB$; take P on AB, so that $AP = AF$. Then P is the dividing point (such that $AB \times PB = AP^2$).

Nils Pipping [14] has recently devised a new construction, in the spirit of Mascheroni and Mohr [11, 12], who proved that every ruler-and-compasses construction can be duplicated with the compasses alone. Pipping's division of the given segment AB requires just seven circles, of three different radii, as in Fig. 2. The circle $A(AB)$ (with center A and radius AB) meets the equal circle $B(AB)$ in two points J and K. Then $J(JK)$ determines L, $B(JK)$ determines M and N, $L(JK)$ determines O, and finally the two circles $M(AO)$ and $N(AO)$ intersect in a point P which divides AB in extreme and mean ratio (as can easily be verified by several applications of Pythagoras's Theorem).

It is interesting to compare this with Mascheroni's third solution to the problem of locating the *mid-point* of a given segment [11, Problem 66], which likewise requires seven circles.

The division into extreme and mean ratio, later known as the *golden section*, was used by Euclid (IV, 10) "to construct an isosceles triangle having each of the angles at the base double of the remaining one" and (IV, 11) "in a given circle to inscribe an equilateral and equiangular pentagon." The figure that he obtained is essentially a regular pentagon with its inscribed star pentagon or *pentagram*. This can be displayed by tying a simple knot in a long strip of paper and carefully pressing it flat. In modern notation, the connection between τ and the pentagon is expressed by the formula

$$\tau = 2 \cos \frac{\pi}{5}.$$

Euclid's construction for the pentagon is one of the thirteen properties of τ described by Fra Luca Pacioli in his book, *Divina proportione* [13] which was illustrated by his friend Leonardo da Vinci. Successive chapters are entitled: The First Considerable Effect; The Second Essential Effect; The Third Singular Effect; The Fourth Ineffable Effect; The Fifth Admirable Effect; The Sixth Inexpressible Effect, and so on. "The Seventh Inestimable Effect" is that a regular decagon of side 1 has circumradius τ. (We can thus inscribe a pentagon in a given circle by first inscribing a decagon and then picking out alternate vertices.) "The Ninth Most Excellent Effect" is that two crossing diagonals of a regular pentagon divide one another in extreme and mean ratio. "The Twelfth Incomparable Effect" and "The Thirteenth Most Distinguished Effect" are constructions for the icosahedron and the dodecahedron. The next chapter tells "how, for the sake of our salvation, this list of effects must end" (because there were just thirteen at table at the Last Supper).

The faces surrounding a corner of the icosahedron belong to a pyramid whose base is a regular pentagon. Any two opposite edges belong to a rectangle whose longer sides are diagonals of such pentagons. Since the diagonal of a pentagon is τ times its side, this rectangle is a *golden* rectangle, whose sides are in the ratio $\tau{:}1$. In fact, the twelve vertices of the icosahedron (Fig.

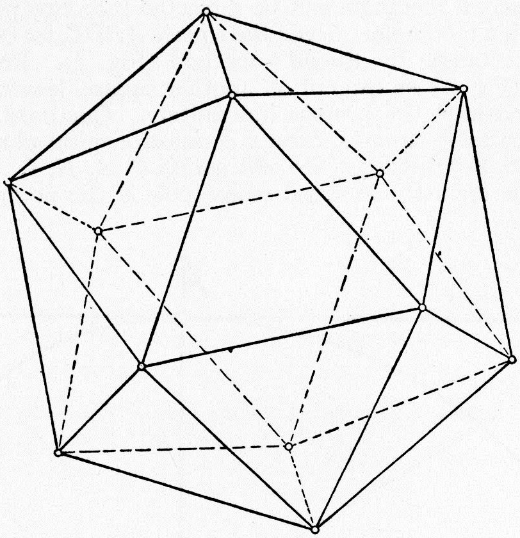

Fig. 3

3) are the twelve vertices of three golden rectangles in perpendicular planes (Fig. 4). Thus [16] the vertices of an icosahedron of edge 2 can be represented by the coordinates

$$(0, \pm 1, \pm \tau), \qquad (\pm \tau, 0, \pm 1), \qquad (\pm 1, \pm \tau, 0).$$

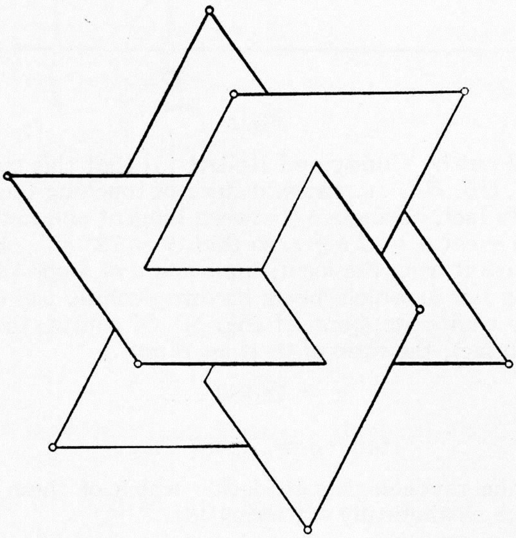

Fig. 4

The identity

$$\tau = 1 + 1/\tau$$

shows that the golden rectangle can be dissected into two pieces: a square and a smaller golden rectangle. Given the square $ABDC$, we can construct the side CF of the rectangle by Euclid's method (Fig. 1). From the smaller rectangle $ABGF$ (Fig. 5) we can cut off another square, leaving a still smaller rectangle, and continue the process indefinitely. Quadrants of circles, inscribed in the successive squares, form a composite spiral of rather agreeable appearance. More interestingly, the end points $D, A, H, I, \ldots$ of the quadrants lie on a true logarithmic spiral whose pole is the point of intersection $CG \cdot BF$.

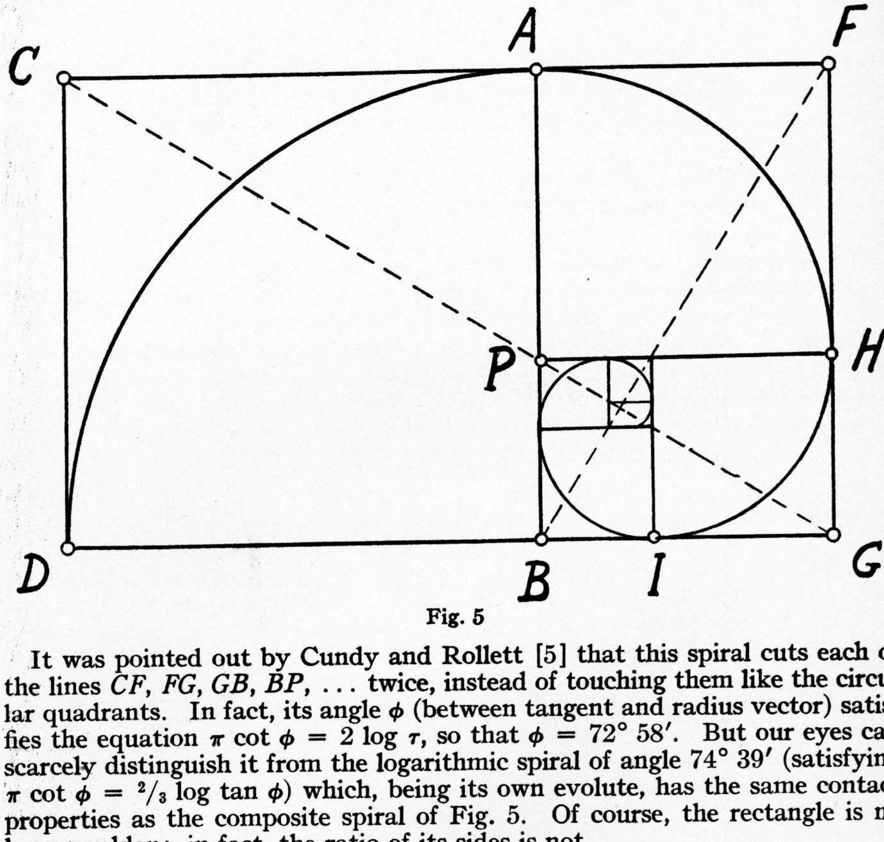

Fig. 5

It was pointed out by Cundy and Rollett [5] that this spiral cuts each of the lines $CF, FG, GB, BP, \ldots$ twice, instead of touching them like the circular quadrants. In fact, its angle ϕ (between tangent and radius vector) satisfies the equation $\pi \cot \phi = 2 \log \tau$, so that $\phi = 72° 58'$. But our eyes can scarcely distinguish it from the logarithmic spiral of angle 74° 39′ (satisfying $\pi \cot \phi = {}^2/_3 \log \tan \phi$) which, being its own evolute, has the same contact properties as the composite spiral of Fig. 5. Of course, the rectangle is no longer golden; in fact, the ratio of its sides is not

$$\tau = 1.6180 \ldots$$

but

$$\tan^{1/3} \phi = 1.5387 \ldots .$$

We leave it to the psychologists to decide which of these two rectangular shapes is the more aesthetically satisfying [6].

In 1202, Leonardo of Pisa, nicknamed Fibonacci (not "son of an ass," as has been suggested, but rather "son of good nature" or "prosperity"), came across his celebrated sequence of integers in connection with the breeding of rabbits [1, 9]. He assumed that rabbits live forever, and that every month each pair begets a new pair which becomes productive at the age of two months. In the first month the experiment begins with a newborn pair of rabbits. In the second month, there is still just one pair. In the third month there are two; in the fourth, three; in the fifth, five; and so on. Let f_n denote the number of pairs of rabbits in the nth month. The first few values may be tabulated as follows:

n:	1	2	3	4	5	6	7	8	9	10	11	12	...
f_n:	1	1	2	3	5	8	13	21	34	55	89	144	...

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Four centuries later, Girard [7] noticed that each of these numbers (after the second) is equal to the sum of the preceding two:

$$f_1 = f_2 = 1, \qquad f_{n+2} = f_{n+1} + f_n \qquad (n \geq 1).$$

Another hundred years passed before Simson [17] observed that f_{n+1}/f_n is the nth convergent to the continued fraction

$$1 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\cdots.$$

To see that this converges to τ, he merely had to express the relation $\tau = 1 + 1/\tau$ in the form

$$\tau = 1 + \frac{1}{\tau} = 1 + \frac{1}{1+}\,\frac{1}{\tau} = 1 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{\tau} = \cdots.$$

Simson also obtained the identity

$$f_{n-1}f_{n+1} - f_n^2 = (-1)^n,$$

which yields the following puzzle-dissection [15]. A rectangle $f_{n-1} \times f_{n+1}$ is cut into four pieces which can apparently be reassembled to form a square of side f_n (Fig. 6). The figure should be drawn on squared paper, so that the

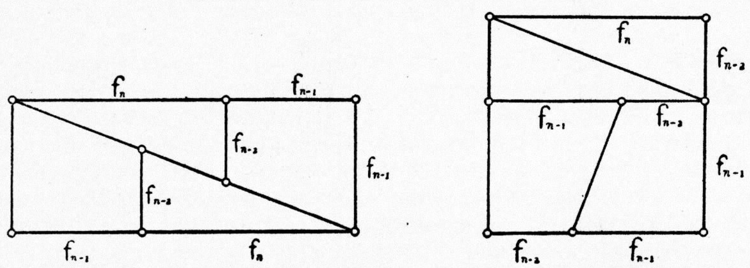

Fig. 6

audience can "see" that there is no cheating. The value $n = 6$ is sufficient in practice, but of course the error is still less detectable when $n = 7$.

Lagrange [8] noticed that the residues of the Fibonacci numbers, for any given modulus, are periodic; e. g., their final digits (in the denary scale) repeat after a cycle of sixty:

$$1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, \ldots, 7, 2, 9, 1, 0.$$ A3893

In 1876, Lucas obtained the identities

$$f_{2n+1} = f_{n+1}^2 + f_n^2, \qquad f_{3n} = f_{n+1}^3 + f_n^3 - f_{n-1}^3,$$

$$1 + 1 + 2 + 3 + \dots + f_n = f_{n+2} - 1.$$

More interestingly [10], he discovered the explicit formula in terms of binomial coefficients:

$$f_{n+1} = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \dots,$$

which can be established by observing that

$$1 + t + 2t^2 + 3t^3 + \dots + f_{n+1}t^n + \dots = (1 - t - t^2)^{-1}$$
$$= 1 + (t + t^2) + (t + t^2)^2 + (t + t^2)^3 + \dots.$$

Setting $t = 0.01$, we obtain the decimal

$$\frac{10000}{9899} = 1.0102030508132134559\dots$$

(which is spoilt by the necessary "carrying" after the nineteenth significant digit).

Lucas also observed that the recursion formula

$$f_{n+2} = f_{n+1} + f_n$$

is satisfied by any linear combination of the nth powers of the roots of the equation

$$x^2 = x + 1,$$

whence, in virtue of the initial conditions $f_0 = 0, f_1 = 1,$

$$f_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$$
$$= 5^{-1/2}\{\tau^n - (-\tau)^{-n}\}.$$

It follows that

$$\tau^n = {}^1/_2(f_n\sqrt{5} + g_n),$$

where

$$g_n = f_{n-1} + f_{n+1}.$$

2. The Fibonacci numbers arise naturally in the botanical phenomenon called *phyllotaxis* [19]. In some trees, such as the elm and lime, the leaves along a twig appear alternately on two opposite sides, and we speak of "$^1/_2$ phyllotaxis." In others, such as the beech and hazel, the passage from one leaf to the next involves a screw-twist through one-third of a turn, and we speak of "$^1/_3$ phyllotaxis." Similarly, the oak and cherry exhibit $^2/_5$ phyllotaxis; the poplar and pear, $^3/_8$; the willow and almond, $^5/_{13}$; and so on. We recognize the fractions as being quotients of alternate Fibonacci numbers. But *consecutive* Fibonacci numbers could be used just as well; e. g., a clockwise rotation through $^5/_8$ of a turn is equivalent to a counterclockwise rotation through $^3/_8$.

Another manifestation of phyllotaxis is the arrangement of the florets of a sunflower, or of the scales of a pine cone, in spiral or helical *whorls*. We observe that the numbers of right-handed and left-handed whorls are two consecutive Fibonacci numbers, viz., 2 and 3 (or vice versa) for the balsam cone, 3 and 5 for the hemlock cone, 5 and 8 for the pine cone, 8 and 13 for the pineapple (the clearest instance of all) and higher numbers for sunflowers of various degrees of cultivation. Church [4] gives photographs of a (34, 55) sunflower and of a giant (55, 89) sunflower. The Russians are said to have succeeded in cultivating a super-giant (89, 144).

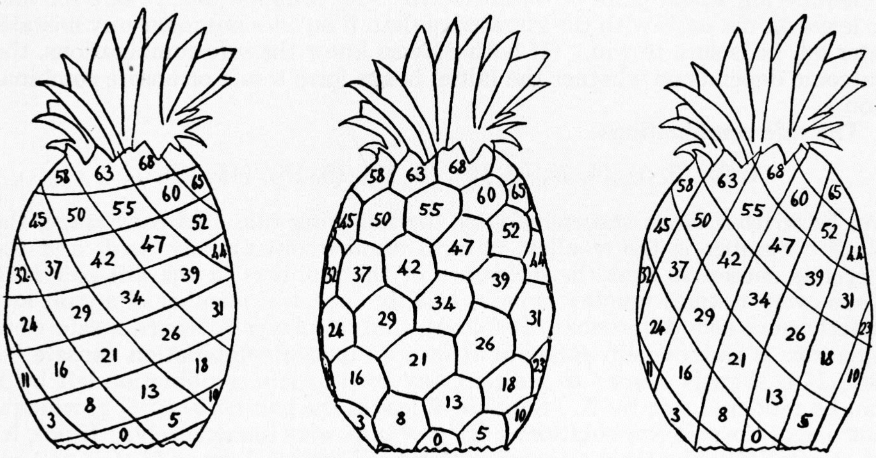

Fig. 7

The fact that the numbers of whorls can be increased by intensive cultivation suggests an evolutionary explanation for the phenomenon. We can imagine that a simple (1, 1) plant evolved into a (1, 2) plant, then into a (2, 3) plant, and so on. The transition can be explained by observing that the florets are not really quadrangular but hexagonal, so that each belongs not only to two kinds of whorl but to a third as well. A slight distortion suffices to make the third kind supersede one of the others. In Fig. 7, a pineapple has been sketched between two hypothetical variants; a simpler fruit, exhibiting (5, 8) phyllotaxis, out of which the pineapple could have evolved, and a super-pineapple, exhibiting unquestionable (8, 13) phyllotaxis, which might be produced by intensive cultivation. The scales of the pineapple have been numbered systematically, with the multiples of 5 and 8 in the directions in which 5 or 8 whorls occur. The remaining numbers then follow by "vector addition," e. g., we have the multiples of $5 + 8 = 13$ in the intermediate direction, in which there are 13 whorls. Thus the numbers in any whorl form an arithmetical progression. The same kind of numbering could be applied to the florets of a sunflower.

Such an explanation for phyllotaxis was first given by Tait [18]. According to Dr. A. M. Turing (who is preparing a new monograph on this subject), the continuous advance from one pair of parastichy numbers to another, such as

(5, 8) to (8, 13), takes place during the growth of a single plant, and may or may not be combined with an evolutionary development.

3. Another application of the golden section is to the theory of *Wythoff's game* [20]. Like the well-known Nim [2], this is a game for two players, playing alternately. Two heaps of counters are placed on a table, the number in each heap being arbitrary. A player either removes from one of the heaps an arbitrary number of counters or removes from both heaps an equal number (e. g., heaps of 1 and 2 can be reduced to 0 and 2, or 1 and 1, or 1 and 0, or 0 and 1). A player wins by taking the last counter or counters.

An experienced player, playing against a novice, can nearly always win by remembering which pairs of numbers are "safe conbinations": safe for him to leave on the table with the knowledge that, if he does not make any mistake later on, he is sure to win. (If both players know the safe combinations, the outcome depends on whether the initial heaps form a safe or unsafe combination.)

The safe combinations A201, A1950

$$(1, 2), (3, 5), (4, 7), (6, 10), (8, 13), (9, 15), (11, 18), \ldots$$

can be written down successively by the following rule. At each stage, the smaller number is the smallest natural number not already used, and the larger is chosen so that the difference of the numbers in the nth pair is n. Thus every natural number appears exactly once as a member of a pair, and exactly once as a difference. It follows that, if player A leaves a safe combination, B cannot help changing it into an unsafe combination (unsafe for B). It is slightly harder to see that any such unsafe combination left by B can be rendered safe by A. Suppose B leaves the pair (p, q) $(p \leq q)$ which is not one of the safe combinations. If $p = q$, A wins immediately. If not, let (p, p') or (p', p) be the safe combination to which p belongs. If $p' < q$, A reduces the q heap to p'. If $q < p'$ (so that $p < q < p'$ and $q - p < p' - p$), he reduces both heaps by equal amounts, so as to leave the safe combination whose difference is $q - p$.

Thus A can win, no matter what B does, unless A is confronted with a safe combination before his first move (in which case he will remove one counter and trust B to make a mistake).

It is easier to write down a lot of safe combinations than to discover a general formula. Such a formula was given by Wythoff "out of a hat"; but a more natural approach is provided by the following theorem of Beatty [3]:

If $x^{-1} + y^{-1} = 1$, where x and y are positive irrational numbers, then the sequences

$$[x], [2x], [3x], \ldots, \qquad [y], [2y], [3y], \ldots$$

together include every positive integer just once.

(Here $[x]$ means the integral part of x.)

The following proof was devised jointly by J. Hyslop in Glasgow and A. Ostrowski in Göttingen.

For a given positive integer N, the numbers of members less than N of the sequences

$$x, 2x, 3x, \ldots \quad \text{and} \quad y, 2y, 3y, \ldots$$

are, respectively, $[N/x]$ and $[N/y]$. Since $x^{-1} + y^{-1} = 1$, where x and y are irrational, N/x and N/y are two irrational numbers whose sum is the integer N. Hence their fractional parts must add up to exactly 1, and

$$[N/x] + [N/y] = N - 1.$$

This is the number of members less than N of the two sequences together. By taking $N = 1, 2, 3, \ldots$ in turn, we deduce that the multiples of x and y are "evenly" distributed among the natural numbers: one between 1 and 2, one between 2 and 3, and so on. Hence their integral parts, $[nx]$ and $[ny]$, are the natural numbers themselves.

This is one of the two requirements for the safe combinations in Wythoff's game. The other, that the difference shall be n, is secured by taking

$$y = x + 1.$$

Since $x^{-1} + y^{-1} = 1$, it follows that

$$x^2 - x - 1 = 0,$$

whence $x = \tau$, $y = \tau^2$, and the nth safe combination is

$$[n\tau], \qquad [n\tau^2].$$

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