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The number of achiral trees*)

By *Frank Harary* and *Robert W. Robinson* at Ann Arbor

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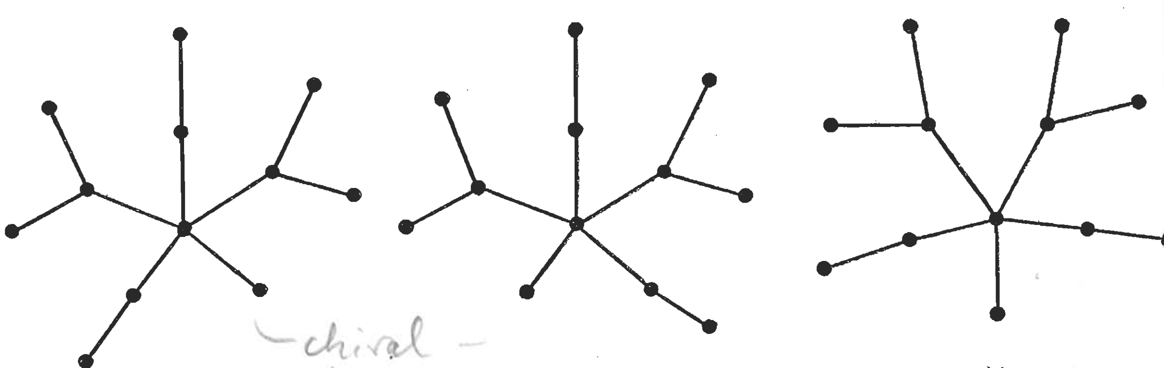
1. Introduction

Cayley [3], the first to count trees, was motivated by the problem of enumerating saturated hydrocarbons. The concept of chirality in graph theory is also suggested by that in organic chemistry. Following [5], p. 102, a *planar graph* can be drawn in the plane with no pair of edges intersecting; a *plane graph* is so drawn. Let G be a plane graph. Then it is self-evident that the reflection of G about any line in the plane results in a unique plane graph G' which may or may not be plane-equivalent to G. If $G' \neq G$, then G is a *chiral graph*; and if $G' = G$, then G is *achiral*. Our first object is to count achiral plane trees which we illustrate in Figure 1. Here trees (a) and (b) are both chiral



(a) (b) (c)

Figure 1. Three different plane trees

and are reflections of each other, while (c) shows an achiral plane tree. Note that these are three different plane trees with the same underlying tree.

A *rooted tree* has a distinguished point called the *root*. A *planted tree* is a rooted tree in which the root is an endpoint. Figure 2 shows three rooted plane trees; two of which are planted, and all have the same underlying tree as that of Figure 1. To illustrate

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chirality for rooted plane trees, we note that in Figure 2, (a) is chiral planted, (b) achiral planted ~~rooted,~~ (c) achiral rooted.

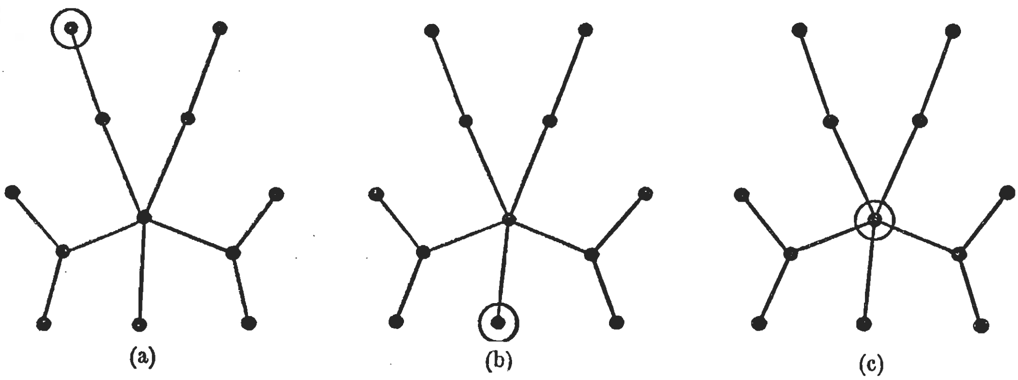

Figure 2. Three rooted plane trees

We count achiral plane trees using the same general method [6], Chap. 3, which works for all known types of unlabeled trees. This procedure counts planted trees (rather easily), then rooted trees in terms of planted ones, and finally trees in terms of the others using the dissimilarity characteristic device originally due to Otter.

This approach also serves to count *partially achiral trees* which have at least one achiral plane embedding and *achiral trees*, for which every plane embedding is achiral. The underlying tree of Figures 1 and 2 is partially achiral but not achiral, while the tree of Figure 3 is achiral, since both of its plane embeddings are achiral.

Figure 3. The plane embeddings of an achiral tree

The number of points is always n and we use the following notation for the number of n-point trees of various types:

Type of tree	Number	Generating function
planted	V_n	$V(x)$
rooted	T_n	$T(x)$
unrooted	t_n	$t(x)$
planted plane	f_n	$f(x)$
rooted plane	g_n	$g(x)$
plane	b_n	$b(x)$
achiral planted plane	p_n	$p(x)$
achiral rooted plane	r_n	$r(x)$
achiral plane	a_n	$a(x)$

Type of tree	Number	Generating function
partially achiral planted	π_n	$\pi(x)$
partially achiral rooted	ϱ_n	$\varrho(x)$
partially achiral	α_n	$\alpha(x)$
achiral planted	P_n	$P(x)$
achiral rooted	R_n	$R(x)$
achiral	A_n	$A(x)$

2. Achiral rooted plane trees

The smallest planted plane trees are shown in Figure 4.

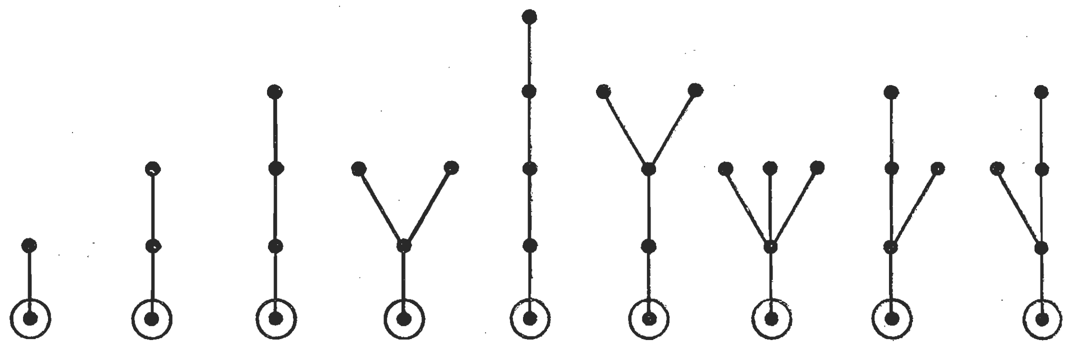

Figure 4. Planted plane trees

We note that $f(x) = \sum_{n=2}^{\infty} f_n x^n$ since a planted tree must have at least two points in order to have an endpoint as the root. It is well-known [7] that

$$f(x) = \frac{x}{2}[1-(1-4x)^{\frac{1}{2}}] \tag{1}$$

and it follows at once from the binomial theorem that

$$f_{n+1} = \frac{1}{n}\binom{2n-2}{n-1}. \tag{2}$$

The numbers in (2) are known as Catalan numbers. Many references on these numbers are given by Brown [2] who traces them back to Euler [4], and Alter [1] supplies even more references.

We now count achiral planted plane trees by expressing $p(x) = \sum_{n=2}^{\infty} p_n x^n$ in terms of itself and of $f(x)$. All the planted plane trees of Figure 4 are achiral except for the last two.

Theorem. *The generating function $p(x)$ for achiral planted plane trees is*

$$p(x) = \frac{x}{2}(-1+(1+2x)(1-4x^2)^{\frac{1}{2}}). \tag{3}$$

so that

$$p_{2n} = \binom{2n-2}{n-1} \quad \text{and} \quad p_{2n+1} = \frac{1}{2}\binom{2n}{n}. \tag{4}$$

Proof. We only need to verify (3) since (4) follows from (3) by the binomial theorem. We first obtain a functional equation for $p(x)$ by analyzing the structure of an achiral planted plane tree T with root r adjacent to point s. Consider the planted plane trees $T_1, T_2, \ldots, T_k$ obtained from T by removing the root r and splitting the point s into k new points, each an endpoint and a root of one of the T_i. These T_i are called the *branches* of T at s and are numbered clockwise from the root in the order in which they occur in the plane. In order to have $T = T'$, its reflection, it is necessary that $T_1 = T'_k$, $T_2 = T'_{k-1}, \ldots$. If k is even, all these trees are accounted for; if k is odd, the remaining planted plane tree $T_{\frac{k+1}{2}}$ must itself be achiral. These considerations suggest the equation with three factors,

$$p(x) = x((px) + x) \sum_{n=0}^{\infty} (f(x^2)/x^2)^n. \tag{5}$$

The first factor x stands for the root point r of T. In the factor $p(x) + x$, the term x stands for the point s in the case of even k while $p(x)$ stands for the tree $T_{\frac{k+1}{2}}$ in the case of odd k. In the last factor, the term $f(x^2)/x^2$ counts the number of ways to choose an ordered pair of planted plane trees which are reflections of each other, and the n'th power counts ordered n-tuples of such choices. The factor $1/x^2$ is to keep from counting the point s more than once.

Equation (5) can be rewritten in the usual way as

$$p(x) = \frac{x(p(x) + x)}{(1 - f(x^2)/x^2)}. \tag{6}$$

Substituting for $f(x^2)$ the expression given in (1) and solving for $p(x)$ gives (3) after routine algebraic manipulation.

The smallest achiral rooted plane trees are shown in Figure 5.

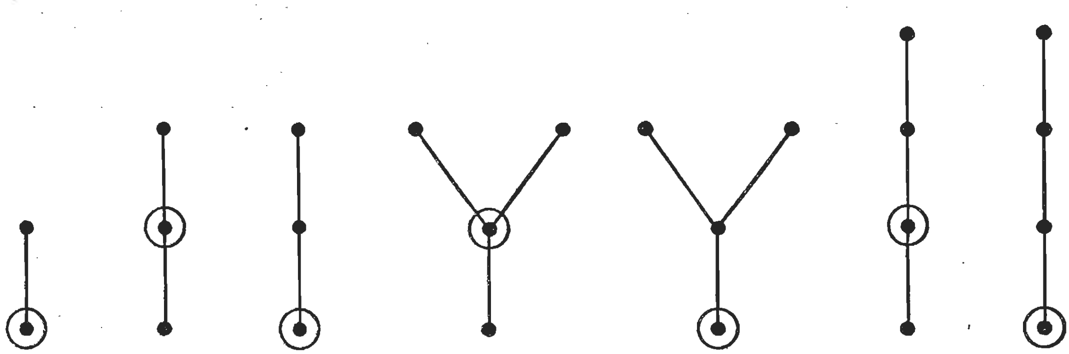

Figure 5. Achiral rooted plane trees

We now count achiral rooted plane trees by expressing $r(x) = \sum_{n=2}^{\infty} r_n x^n$ in terms of $f(x)$ and $p(x)$, which requires the use of Pólya's counting theorem.

Theorem. *The generating function $r(x)$ for achiral rooted plane trees is*

$$r(x) = \frac{x^2}{1 - 2x} \tag{7}$$

so that

$$r_n = 2^{n-2}. \tag{8}$$

Proof. Relation (8) is immediate from (7). A functional equation for $r(x)$ is obtained by analyzing the structure of an achiral rooted plane tree T with root r. Consider the branches of T at r, say $T_1, \ldots, T_k$ in clockwise order around r as they occur in the plane. Since the starting point is arbitrary, the sequence $T_{\sigma(1)}, \ldots, T_{\sigma(k)}$ corresponds to the same tree T for any cyclic permutation σ of $\{1, \ldots, k\}$. Since the converse is evident, we have (in the parlance of Pólya's counting theorem) established a 1 — 1 correspondence between rooted plane trees and configurations from a domain with cyclic group into the set of planted plane trees.

In order to apply Pólya's classical enumeration theorem, we use the cycle index $Z(C_k; s_1, s_2, \ldots, s_k)$ of the cyclic group C_k which is a polynomial in the k variables s_i; see Pólya [9] or [6], Chap. 2. Then the generating function for rooted plane trees, which we require as an intermediate step is

$$g(x) = x \sum_{k=1}^{\infty} Z(C_k; f(x)/x, f(x^2)/x^2, \ldots, f(x^k)/x^k). \tag{9}$$

This result was obtained in [7], in connection with the enumeration of plane trees.

Now suppose that instead of the cyclic group on $\{1, \ldots, k\}$ we allow the dihedral group D_k, which includes in addition the k reflections. It is well known [5], p. 184, that

$$(10)\quad \begin{cases} Z(D_{2k+1}; s_1, \ldots, s_{2k+1}) = \frac{1}{2} Z(C_{2k+1}) + \frac{1}{2} s_1 s_2^k \\ Z(D_{2k}; s_1, \ldots, s_{2k}) = \frac{1}{2} Z(C_{2k}) + \frac{1}{4} s_2^k + \frac{1}{4} s_1^2 s_2^{k-1}, \end{cases}$$

in which it is understood that $Z(C_n) = Z(C_n; s_1, \ldots, s_n)$.

However we need a modification of this cycle index in order to include a new variable s_1^* which indicates a point fixed by a reflection. Then the new cycle index which results is written $Z(D_m; s_1, s_2, \ldots, s_m; s_1^*)$ and is obtained by replacing s_1 by s_1^* in the last term of each equation in (10).

Using this modification of $Z(D_m)$, the following generating function counts isomorphism classes of rooted plane trees with respect to both rotations and reflections:

$$x \sum_{m=1}^{\infty} Z(D_m; f(x)/x, f(x^2)/x^2, \ldots, f(x^m)/x^m; p(x)/x). \tag{11}$$

Strictly speaking, a complete proof would have to begin with Burnside's Lemma, [5], p. 181.

The difference between (9) and (11) is that any tree T is considered equivalent to its reflection T' under the dihedral group, so the pair is counted only once in (11) instead of twice as in (9). Thus the rooted plane trees which are achiral are counted by

$$\sum_{m=1}^{\infty} [2x\, Z(D_m; f(x)/x, \ldots, f(x^m)/x^m; p(x)/x) - xZ(C_m; f(x)/x, \ldots, f(x^m)/x^m)]. \tag{12}$$

In view of the modification of (10) to contain the new variable s_1^*, this becomes

$$\sum_{m=1}^{\infty} \left[p(x) f(x^2)^m/x^{2m}) + \left(\frac{1}{2} f(x^2)^m/x^{2m-1}\right) + \left(\frac{1}{2} p(x)^2 f(x^2)^{m-1}/x^{2m-1}\right)\right]. \tag{13}$$

Rewriting (13) in the usual way, we obtain

$$r(x) = \frac{x}{2}\left(1 + \frac{p(x)}{x}\right)^2 \left(1 - \frac{f(x^2)}{x^2}\right)^{-1}. \tag{14}$$

With the help of (6), we easily find

$$r(x) = \frac{p^2(x)}{2x^2} + \frac{p(x)}{2x} - \frac{x}{2}. \tag{15}$$

Upon substituting for $p(x)$ the expression given in (3), the relation simplifies to (7).

We note in passing that the reasoning which led to (12) is analogous to the key idea of Read [10] for counting self-complementary graphs and digraphs.

3. Dissimilarity characteristic

Because the result (16) will be used in counting achiral plane trees, partially achiral trees, and (totally) achiral trees, we develop it by itself in this lemma-like section.

In the now classical dissimilarity characteristic equation for trees due to Otter [8], the equation $p - q = 1$ for a given tree T with p points and q lines [5], p. 33 is recast into a new form which counts points and lines with equivalence determined by the automorphism group $\Gamma(T)$. We now require a further modification of Otter's equation in which there is a given tree T and some subtree S which is invariant under all automorphisms of T.

Let p^* be the number of points of S which are dissimilar under the action $\Gamma(T)$, q^* the corresponding number of dissimlar lines of S, and $s^* = 1$ or 0 according as T has or has not a "symmetry line" which is mapped to itself with ends reversed by some automorphism. Then the subtree form of Otter's dissimilarity characteristic equation is

$$p^* - q^* + s^* = 1. \tag{16}$$

The proof is omitted, as it is essentially the same as that of Otter's equation, out-[l]ined in [5], p. 189.

The different applications of (16) are obtained by considering appropriate subtrees S of T. The purpose in each case is to obtain the number of unrooted trees of the desired kind in terms of rooted trees. For achiral plane trees, we will take the subtree S consisting of those points and lines left fixed by some reflection of T, as we show in the next section.

4. Achiral plane trees

In Sections 1 and 2 we found explicit closed forms (1), (3) and (7) for the ordinary generating functions $f(x)$, $p(x)$ and $r(x)$ which count plane trees which are planted, achiral planted and achiral rooted, respectively. Using (16) we now find $a(x)$ in terms of $f(x)$, $p(x)$ and $r(x)$. For any achiral plane tree T let T^* be the subgraph consisting of all points and lines of T which are left fixed by some reflection of T. For a line to be included in T^* we insist that some reflection of T fix both of its points. It is clear that T^* is invariant under all the automorphisms of T as a plane graph. It is to the subgraphs T^* that (16) will be applied.

We now see that if T^* is not empty, then it is a tree. Since T is a tree, the subgraph T^* is a tree just if it is connected. It is well known [5], p. 35, that T has an invariant called its *center*, consisting either of a single point or two adjacent points. If the center is a single point c, then c is fixed under every reflection of T. Consequently c is in T^*. If u is any other point in T^*, then every point and line on the unique path joining c and u is in T^* as well, so T^* is indeed a tree. The same reasoning applies if the center of T is a pair of points c, c' which are both fixed under some reflection of T. For if c

and c' are interchanged by a reflection ϱ of T, it is clear that no point or line is left fixed by ϱ, and so ϱ plays no part in the formation of T^*. Thus the only case in which T^* is not a tree is when no reflection of T leaves any point fixed, in which case of course T^* is empty.

We now illustrate (16) for the plane tree T of Figure 6(a) and its reflection-invariant subtree T^* of Figure 6(b). Here T has 8 points and T^* has 4, but $p^* = 3$ since equivalence is taken according to the group of T. Obviously $q^* = 2$ and $s^* = 0$ here.

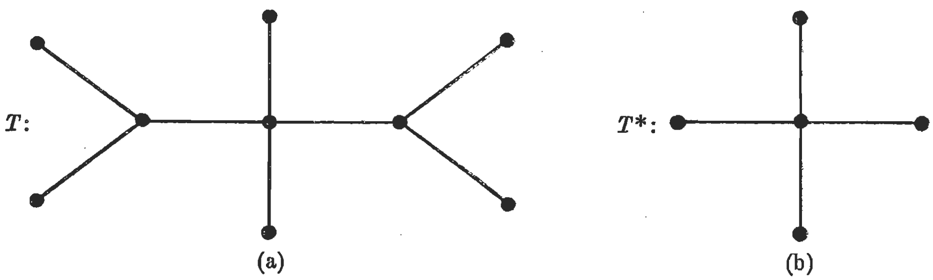

Figure 6. A plane tree and its reflexion-invariant subtree

For any achiral plane tree T for which T^* is nonempty, let $p^*(T)$, $q^*(T)$ and $s^*(T)$ be the p^*, q^* and s^* in equation (16) for the subtree $S = T^*$. If T^* is empty, let $p^*(T) = q^*(T) = 0$ and $s^*(T) = 1$. Then (16) becomes

$$1 = p^*(T) - q^*(T) + s^*(T) \tag{7}$$

if T^* is not empty; our convention assures that (17) also holds in case T^* is empty. Thus $s^*(T) = 1$ while $p^*(T) = q^*(T) = 0$.

Now consider the effect of summing (17) over all achiral plane trees T with n points. On the left we have

$$\Sigma\, 1 = a_n. \tag{18}$$

On the right, the first term of (17) gives

$$\Sigma\, p^*(T) = r_n. \tag{19}$$

To see this, for each achiral plane tree T and each point u of T^* let T_u be the result of rooting T at u. Since u is in T^*, T_u is achiral because it is mapped to itself by any reflection of T which leaves u fixed. It is easy to see that every achiral rooted tree is obtained in this way. Moreover $T_u \cong T_v$ implies that u and v are in the same equivalence class with respect to $\Gamma(T)$. Restricting attention to trees on n points, this gives a $1-1$ correspondence which justifies (19).

The sum $\Sigma\, q^*(T)$ can be similarly evaluated, but the result is most easily expressed as a generating function. For any tree T, let $n(T)$ be the number of points of T. Summing over *all* achiral plane trees T we have

$$\Sigma\, x^{n(T)} = a(x), \tag{20}$$

$$\Sigma\, p^*(T) x^{n(T)} = r(x) \tag{21}$$

directly from (18) and (19).

To evaluate $\Sigma\, q^*(T) x^{n(T)}$, start with any achiral plane tree T and any line (u, v) in T^*. Form the pair of planted plane trees (T^u, T^v) by taking for T^u the union of the line (u, v) rooted at v with the connected component of $T - (u, v)$ which contains u, and T^v similarly. Because (u, v) is a line in T^*, it is easy to see that T^u and T^v are both achiral. All pairs of achiral planted plane trees are obtained in this manner, and the unordered pair (T^u, T^v) corresponds uniquely to the tree T and the unordered pair of points (u, v). Clearly, unordered pairs of achiral planted plane trees are counted by $\frac{1}{2}(p^2(x) + p(x^2))$ with the total number of points as the enumeration parameter. Since two extra points are added in the formation of (T^u, T^v) from T, we see that

$$\Sigma q^*(T) x^{n(T)} = \frac{1}{2}\left(\frac{p^2(x)}{x^2} + \frac{p(x^2)}{x^2}\right). \tag{22}$$

Similarly it is not hard to establish that if $s^*(T) = 1$ for an achiral plane tree T, then T has two central points (u and v) and $(T^u)' = T^v$. Conversely any unordered pair of reflection-dual planted plane trees corresponds uniquely to an achiral plane tree with a symmetry line. The planted plane trees which are self-dual under reflection are by definition just the achiral ones, so we find

$$\Sigma s^*(T) x^{n(T)} = \frac{1}{2}\left[\frac{f(x^2)}{x^2} + \frac{p(x^2)}{x^2}\right]. \tag{23}$$

Combining (20) and (21) with (22) and (23), and applying these to (17), we obtain the following equation after a bit of primitive manipulation,

$$a(x) = r(x) - \frac{1}{2}\left[\frac{p^2(x)}{x^2} - \frac{f(x^2)}{x^2}\right]. \tag{24}$$

which counts achiral plane trees in terms of $r(x)$, $p(x)$, and $f(x)$. On replacing $r(x)$ in (24) by its expression in (15), we obtain

$$a(x) = \frac{1}{2}\left[-x + \frac{p(x)}{x} + \frac{f(x^2)}{x^2}\right]. \tag{25}$$

Finally, by (1) and (3) and just a little algebraic manipulation, we derive the remarkable and surprising result,

$$a(x) = p(x), \tag{26}$$

whose coefficients can be obtained explicitly from (4).

5. Partially achiral trees

Recall that a tree is partially achiral if it has a plane embedding which is an achiral plane tree. As in the case of achiral plane trees, we count planted and rooted ones first, and then use the dissimilarity characteristic equation (16) to find the unrooted ones. The ordinary generating function $T(x)$ for rooted trees is well known from Pólya [9]; also [5], p. 187:

$$T(x) = x \exp \sum_{m=1}^{\infty} T(x^m)/m. \tag{27}$$

We denote by $\pi(x)$, $s(x)$, $\alpha(x)$ respectively, as in Section 1, the ordinary generating functions for partially achiral trees which are planted, rooted, and unrooted. The results are summarised in the next three equations;

$$\pi(x) = \left(1 + \frac{\pi(x)}{x}\right) T(x^2), \tag{28}$$

$$\varrho(x) = \frac{T(x^2)}{x^2}\left(x + \pi(x) + \frac{1}{2x}(\pi^2(x) - \pi(x^2))\right), \tag{29}$$

$$\alpha(x) = \varrho(x) - \frac{1}{2x^2}(\pi^2(x) + \pi(x^2)) + T(x^2). \tag{30}$$

We now proceed to prove them. For (28), we need to consider the structure of a planted partially achiral tree, rooted at point u adjacent to point v. When u is removed, consider the remaining b branches at v. Either b is even and the b branches can be divided into two identical rooted trees, or else b is odd, so that one branch cannot be paired off and must itself be partially achiral.

The pair of identical rooted trees give the factor $T(x^2)$ while the either-or consideration accounts for the other factor in (28).

For (29), consider the structure of a partially achiral rooted tree in terms of the b branches at the root, each regarded as planted at the root. If b is odd these branches must form a pair of identical rooted trees, along with a single branch which must be partially achiral. If b is even, we may find either a pair of identical rooted trees or there may be two additional branches which cannot be paired off, and each of these must be partially achiral. As before, the factor $T(x^2)$ counts pairs of identical rooted trees. In the other factor of equation (29), there are three terms. The term $\pi(x)$ accounts for the single unpaired branch in case b is odd; the x accounts for the root point in case b is odd and all branches occur in pairs; and $\frac{1}{2x}(\pi^2(x) - \pi(x^2))$ accounts for the unordered pairs of disctinct partially achiral branches in the remaining case.

The proof of (30) is a bit more involved. Consider now a random partially achiral plane tree R (for random). Let the achiral plane trees with underlying tree R be $R_1, \ldots, R_k$, and let $R^* = R_1^* \cup \cdots \cup R_k^*$, where R_i^* consists as in Section 4 of the points and lines left fixed by a reflection of R_i. As seen in Section 4, each R_i^* is either empty or else is a tree containing the center of R, and so R^* also must be empty or else a tree. Evidently R^* is invariant under all the automorphisms of R, that is under $\Gamma(R)$. Let p^* and q^* be the number of dissimilar points and lines of R^* under $\Gamma(R)$, and let s^* be 1 or 0 according to whether or not R has a symmetry line. Then if R^* is not empty the dissimilarity characteristic equation (16) holds.

If R^* is empty let T be some plane embedding of R which is achiral. Since T^* must be empty, the (unique) reflection of T leaves no point fixed, so that the center of T must be a pair of points which are $\Gamma(R)$-similar. Thus $p^* = q^* = 0$ and $s^* = 1$ in this case, so (16) holds for every partially achiral tree. To obtain a relation for $\alpha(x)$, we sum (16) over all partially achiral trees, weighted with the factor $x^{n(R)}$, with $n(R)$ the number of points of R. As soon as we have justified the equations

$$\Sigma\, x^{n(R)} = \alpha(x) \tag{31}$$

$$\Sigma\, p^*(R) x^{n(R)} = \varrho(x) \tag{32}$$

$$\Sigma\, q^*(R) x^{n(R)} = \frac{1}{2x^2}(\pi^2(x) + \pi(x^2)) \tag{33}$$

$$\Sigma\, s^*(R) x^{n(R)} = T(x^2), \tag{34}$$

we will have proved that (30) holds. Clearly (31), (32) and (33) follow by reasoning similar to that for (20), (21) and (22). The trees with a symmetry line are enumerated by $T(x^2)$, since such a tree is obtained in exactly one way by taking two copies of a rooted tree and joining the two root points. Such trees are always partially achiral, which justifies (34).

Solving equation (28) for $\pi(x)$, we get an explicit expression for this generating function,

$$\pi(x) = \frac{T(x^2)}{1 - T(x^2)/x}. \tag{35}$$

6. Achiral trees

Recall that a tree is achiral if every plane embedding is achiral. As usual we first count planted ones, then rooted, and finally unrooted. The generating functions are denoted by $P(x)$, $R(x)$, and $A(x)$ respectively, as in Section 1. We will show that $P(x)$ satisfies

$$P(x) = x^2\left(1 + \sum_{n=1}^{\infty} P(x^n)/x^n\right). \tag{36}$$

To see this, let S be any planted tree with root point u adjacent to point v. Consider the branches of $S - u$ at v, planted at v. In order for S to be achiral, it is necessary and sufficient for all of these branches to be isomorphic and themselves achiral, and for each to have just one plane embedding if there be more than one branch. The reason is that embedding S amounts to embedding these branches in some order, that is, the branches are ordered arbitrarily and then each is embedded in the plane arbitrarily. The effect of a reflection is to reverse the order of the branches and to reflect each one. Hence if there are two branches with distinct plane embeddings, then we can easily find an embedding of S which is not an achiral plane tree.

By induction on the size of achiral planted trees, it follows that each has just one plane embedding; the converse is obvious. Thus the conditions just amount to the branches being achiral and isomorphic. Sets of n isomorphic achiral planted trees are counted by $P(x^n)$. In (36) we divide by x^n to account for the identification of the root points, and multiply in the end by x^2 to restore the original root point and its neighbor. Of course n runs from 1 to ∞, and the term 1 is added for the 2-point planted tree.

The equation which we found to express $R(x)$ in terms of $P(x)$ is

$$R(x) = 2x + \frac{P^2(x)}{x^2} + (1 - x)\frac{P(x)}{x}\left(\frac{P(x^2)}{x^2} - 1\right) - \left(\frac{P^2(x) - P(x^2)}{2x} + \frac{P(x^3)}{x^2}\right) - \left(\frac{P^2(x^2) - P(x^4)}{2x^3}\right). \tag{37}$$

Let $R^{(n)}(x)$ count these achiral rooted trees with exactly n branches at the root, so that

$$R(x) = \sum_{n=0}^{\infty} R^{(n)}(x). \tag{38}$$

The behavior of $R^{(n)}(x)$ is irregular for $n < 5$:

$$R^{(0)}(x) = x, \tag{39}$$

$$R^{(1)}(x) = P(x), \tag{40}$$

$$R^{(2)}(x) = \frac{P^2(x) + P(x^2)}{2x}, \tag{41}$$

$$R^{(3)}(x) = \frac{P(x)P(x^2)}{x^2}, \tag{42}$$

$$R^{(4)}(x) = \frac{P(x)P(x^3)}{x^3} + \frac{P^2(x^2) - P(x^4)}{2x^3}, \tag{43}$$

$$R^{(n)}(x) = \frac{P(x^{n-2})P(x^2) + P(x^{n-1})P(x) - P(x^n)}{x^{n-1}}, \tag{44}$$

the latter valid for $n \geq 5$.

Now (37) is obtained from (38)—(44) with three applications of (36) to simplify the infinite sums. Of these, equations (38)—(40) are obvious. We give briefly the considerations which lead to the remaining four relations (41)—(44).

A plane embedding of a rooted tree amounts to a cyclic ordering of its branches at the root followed by plane embeddings of each of these branches. Any reflection which leaves the root fixed simply reverses the cyclic order on the branches and reflects each branch.

Now every achiral rooted tree must satisfy:

(i) Every branch is achiral.

(ii) There are at most two isomorphism classes of branches.

(iii) If there are two isomorphism classes of branches, one class consists of no more than two branches.

For if one branch is chiral, then a chiral embedding of the rooted tree is obtained by choosing the same chiral embedding for all branches isomorphic to this one.

Second, if there are more than two isomorphism classes of branches, then a chiral embedding of the rooted tree is obtained by assigning the branches in the same isomorphism class consecutive places in the cyclic order around the root.

Finally, if there are two isomorphism classes of branches with at least three members each, let all but one member of one of these classes be assigned consecutive places in the cyclic order around the root, and the last member displaced from this group by one position. Any such embedding is chiral.

With the help of (i), (ii) and (iii) we are ready to justify equations (41)—(44). Equation (41) for $R^{(2)}(x)$ considers rooted trees with just two branches, which are of course interchangeable achiral planted trees. It is quite well known that unordered pairs are counted as in (41), while the x in the denominator corrects dimensionally for the root point. There is a similar factor in the denominator of (42)—(44) for the same reason.

For (42), an achiral rooted tree with three branches must have two alike, whence $P(x^2)$, whilst the third gives $P(x)$. The first term of (43) similarly handles 4-branched rooted trees with at least 3 like branches, whereas the second term counts two pairs different ones.

In general, an achiral rooted tree with n branches either has all branches of one kind, or has one sort of branch represented exactly twice or once. The latter possibilities are counted by the first two terms on the right of (44). But each of these terms also counts trees for which all the branches are alike, hence the inclusion-exclusion type of correction in the third term.

Now that rooted achiral trees have been enumerated in (37), we are ready to handle the unrooted ones, by verifying:

$$A(x) = R(x) - \frac{1}{2x^2}(P^2(x) - P(x^2)). \tag{45}$$

This again utilizes the dissimilarity characteristic approach.

If S is achiral and $S_1, \ldots, S_k$ are the plane embeddings of S, let

$$S^{(*)} = S_1^* \cap \cdots \cap S_k^*.$$

Thus $S^{(*)}$ contains those points and lines fixed by some reflection of every embedding of S in the plane. Let the number of S-dissimilar points and lines in $S^{(*)}$ be $p^{(*)}$ and $q^{(*)}$. Also let $s^{(*)} = 0$ or 1, the latter just if S contains a symmetry line.

Since an intersection of connected graphs is connected, we know that $S^{(*)}$ is a tree if it is not empty. When we sum the terms of (16) over all achiral trees S, weighted by the number $n(S)$ of points, we find analogously as before that

$$\sum_S x^{n(S)} = A(x), \tag{46}$$

$$\sum_S p^{(*)}(S) x^{n(S)} = R(x), \tag{47}$$

$$\sum_S q^{(*)}(S) x^{n(S)} = \frac{1}{2x^2}(P^2(x) + P(x^2)), \tag{48}$$

$$\sum_2 s^{(*)}(S) x^{n(S)} = \frac{P(x^2)}{x^2}. \tag{49}$$

On combining (46)—(49) as directed by (16), we obtain the desired equation (45) for achiral trees.

7. Data

We present in Table 1 the number of trees (for $n = 1$ to 12 points) of all the types studied in this paper. It is essential to keep in mind the notation for all 15 kinds of trees listed at the end of Section 1. The table has three column-categories giving planted, rooted, and unrooted trees. For each of these categories, the five kinds of trees enumerated are (in order) plane, achiral plane, ordinary, partially achiral, and achiral. The three kinds of trees (from among these five) dealing with chirality were counted using equations developed in preceding sections. Plane trees were counted in [7], where the equations for the generating functions are derived, but the numbers supplied only through $n = 7$. Ordinary trees go back to Cayley [3], Pólya [9], and Otter [8].

Table 1. Three numbers

	Planted					Rooted					Unrooted				
n	f_n	p_n	v_n	π_n	P_n	g_n	r_n	T_n	ϱ_n	R_n	h_n	a_n	t_n	α_n	A_n
1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	1	1	1	1	1	2	2	2	2	2	1	1	1	1	1
4	2	2	2	2	2	4	4	4	4	4	2	2	2	2	2
5	5	3	4	3	3	10	8	9	8	8	3	3	3	3	3
6	14	6	9	6	5	26	16	20	16	15	6	6	6	6	6
7	42	10	20	10	6	80	32	48	31	26	14	10	11	9	9
8	132	20	48	19	10	246	64	115	62	45	34	20	23	19	16
9	429	35	115	33	11	810	128	286	120	71	95	35	47	30	23
10	1430	70	286	62	16	2704	256	719	236	110	280	70	106	61	35
11	4862	126	719	110	19	9252	512	1842	454	168	854	126	235	99	51
12	16796	252	1842	204	26	32066	1024	4766	904	247	2694	252	551	208	72

8. Unsolved problems

Recall that f_n gives the number of planted plane trees and that p_n, r_n and a_n are numbers of achiral plane trees which are planted, rooted and unrooted. We derived explicit formulas for f_n, p_n and r_n in (2), (4) and (8) and showed in (26) that $a(x) = p(x)$. These results imply the following relations.

$$r_{n+1} = 2r_n, \tag{50}$$

$$p_{2n} = 2p_{2n-1} = nf_{n+1}, \tag{51}$$

$$a_n = p_n, \tag{52}$$

which establish various $1-1$, $2-1$, and $n-1$ correspondences. The only one to be explained so far by a natural structural correspondence is

$$p_{2n} = 2a_{2n-1}. \tag{53}$$

This can easily be verified by proving that every achiral plane tree on an odd number of points can be made into an achiral planted plane tree by the addition of a root point in just two ways. The same is true of any achiral plane tree with no symmetry line. However if T has the symmetry line (u, v), then there is just one way of obtaining an achiral planted plane tree from T if T^u is achiral, and none at all if T^u is chiral. These facts allow (25) to be derived directly without recourse either to the dissimilarity characteristic or to the evaluation of $r(x)$.

1. What other explicit correspondences can be found relating to (50), (51) and (52)?

2. Asymptotic formulas for those numbers for which we already have explicit formulas are superfluous. Pólya [9] and Otter [8] determined asymptotic formulas for rooted trees and ordinary trees. Their methods and results are applicable to finding asymptotic expressions for π_n, ϱ_n and α_n, so there is really no problem here, so far. The asymptotic enumeration for plane trees was left open in [7] and has not yet been done.

Thus the new problem which we now propose is to provide asymptotic formulas for achiral trees.

3. Not only have achiral plane trees been defined, but also achiral plane graphs. What is the number of achiral plane graphs with a given number of vertices and edges?

In particular, the number of achiral plane unicyclic graphs can be easily derived from the results in this paper since it involves combinations of achiral rooted plane trees.

4. How many trees with n points have a unique plane embedding? In general, how many have just m plane embeddings, $m = 2, 3, \ldots$? One may also ask for the number of plane embeddings of a given abstract tree.

5. As noted in the books [5] and [6], labeled enumeration is usually much more tractable than the unlabeled case. However, we have counted achiral trees of various kinds which are unlabeled. What are the corresponding formulas for these achiral trees when labeled?

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Mathematics Department, University of Michigan, Ann Arbor, MI 48104, USA

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